Math 484: Nonlinear programming

Chapter 1: Lecture 1

Important course information

- The first homework assignment is due on Wednesday, 1-24
- The course webpage is hosted on Canvas, at canvas.illinois.edu
- Homework assignments and solutions, as well as quiz and test grades, will be posted on Canvas
- Lecture slides will be posted on Canvas, but you will need to come to class to learn the course content

What is nonlinear programming?

Programming = optimization

$$\begin{aligned} \min_{(x_1,\ldots,x_n)\in\mathbb{R}^n} \quad & f(x_1,\ldots,x_n) \\ \text{subject to} \quad & g_1(x_1,\ldots,x_n) \le 0 \\ & \vdots \\ & g_m(x_1,\ldots,x_n) \le 0 \end{aligned}$$

Today

$$\min_{x \in \mathbb{R}} \quad f(x)$$

Definitions

Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. We say that a value $x^* \in \mathbb{R}$ is a *global minimizer* if $f(x^*) \leq f(x)$ for all $x \in D$.

Examples, with $D = \mathbb{R}$:

- $f(x) = (x-2)^2 + 3$
- $f(x) = -(x-2)^2 + 3$
- $f(x) = 0$

Definitions

Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. We say that a value $x^* \in \mathbb{R}$ is a *local minimizer* if there exists a value $r > 0$ such that $f(x^*) \leq f(x)$ for all $x \in (x^* - r, x^* + r) \cap D$.

Examples, with $D = \mathbb{R}$:

- $f(x) = \sin x$
- $f(x) = \lfloor x \rfloor$

Definitions

We say that x^* is a *strict* (global or local) minimizer if the inequality in the definition is strict. In other words, $f(x^*) < f(x)$ for all relevant $x \neq x^*$.

We can define *maximizers* by flipping the inequality symbols in all previous definitions.

Existence of global minimizers

Theorem (Extreme value theorem)

Let $a \leq b$ be real numbers. If $f : [a, b] \to \mathbb{R}$ is a continuous function, then f has a global minimizer $x^* \in [a, b]$.

Existence of global minimizers

A function $f : \mathbb{R} \to \mathbb{R}$ is *coercive* if

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty.$$

Proposition

If $f : \mathbb{R} \to \mathbb{R}$ is coercive and continuous, then f has a global minimizer.

Write $f(0) = c$. As $\lim_{x \to -\infty} f(x) = \infty$, there exists $a \in \mathbb{R}$ such that for all $x < a$, $f(x) > c$. Similarly, there exists $b \in \mathbb{R}$ such that for all $x > b$, $f(x) > c$. Observe that $f(x) > c$ for all $x \notin [a, b]$, so $0 \in [a, b]$.

By the extreme value theorem, f has a global minimizer x^* when the domain of f is restricted to $[a, b]$. We claim that x^* is a global minimizer of f over the domain $\mathbb{R}$.

We aim to show that $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}$. If $x \in [a, b]$, then $f(x) \leq f(x^*)$, as x^* is a global minimizer over the domain $[a, b]$. If $x \notin [a, b]$, then as $f(x^*) \leq f(x')$ for all $x' \in [a, b]$, it follows that

$$f(x^*) \leq f(0) = c < f(x).$$

Hence, x^* is a global minimizer of f.

First derivative test

Theorem (Theorem 1.1.4)

Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is a differentiable function and x^* is a local minimizer, then $f'(x^*) = 0$, or x^* is an endpoint of I.

- Why do we need to assume differentiability?
- Why is this not a biconditional?

First derivative test

Theorem (Theorem 1.1.4)

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- Why do we need to assume differentiability?
- Why is this not a biconditional?

If $f'(x^*)$ exists and equals 0, then x^* is a *critical point* of f.

Theorem (Theorem 1.1.5)

If $f : \mathbb{R} \to \mathbb{R}$ is a function with a continuous second derivative and x^* is a critical point of f, then:

1. *If $f''(x) \geq 0$ for all $x \in \mathbb{R}$, then x^* is a global minimizer on $\mathbb{R}$.*
2. *If there exists an interval $[a, b]$ containing x^* so that $f''(x) \geq 0$ for all $x \in [a, b]$, then x^* is a global minimizer on $[a, b]$.*
3. *If $f''(x^*) > 0$ then x^* is a local minimzer.*

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By Taylor's theorem, for each $x \in \mathbb{R}$, there exists a value ζ between x and x^* such that

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2}f''(\zeta)(x - x^*)^2.$$