Math 484: Nonlinear programming

Chapter 1: Lecture 4

Definite matrices

A symmetric $n \times n$ matrix A is:

- *positive definite* if $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$
- *positive semidefinite* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- *negative definite* if $\mathbf{x}^T A\mathbf{x} < 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$
- *negative semidefinite* if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$

We say that A is *indefinite* if none of these apply.

Quadratic forms

The *quadratic form* of a symmetric matrix A is a function $f : \mathbb{R}^n \to \mathbb{R}$ with the form $\mathbf{x} \mapsto \mathbf{x}^T A \mathbf{x}$.

We say that f is *positive/negative (semi)definite* if and only if A is positive/negative (semi)definite.

Quadratic forms

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The function f is a quadratic form of some symmetric matrix A if and only if f has a polynomial representation in which each nonzero term has degree 2.

Restating a previous theorem

Theorem

Let $D \subseteq \mathbb{R}^n$ be an open set. Let $f : D \to \mathbb{R}$ be a function, and suppose $\mathbf{x}^* \in D$ is a critical point of f. If there exists a value $r > 0$ such that

$$\mathbf{u}^T Hf(\mathbf{x})\, \mathbf{u} \geq 0 \textit{ for all } \mathbf{u} \in \mathbb{R}^n \textit{ and } \mathbf{x} \in B(\mathbf{x}^*, r),$$

then $\mathbf{x}^*$ is a local minimizer of f.

Restating a previous theorem

Theorem

Let $D \subseteq \mathbb{R}^n$ be an open set. Let $f : D \to \mathbb{R}$ be a function, and suppose $\mathbf{x}^* \in D$ is a critical point of f. If there exists a value $r > 0$ such that

$$Hf(\mathbf{x}) \textit{ is positive semidefinite for all } \mathbf{x} \in B(\mathbf{x}^*, r),$$

then $\mathbf{x}^*$ is a local minimizer of f.

Characterizing definite matrices

Theorem (Spectral theorem)

A matrix A is symmetric if and only if there exists an orthogonal matrix Q and a diagonal matrix Λ such that

$$A = Q\Lambda Q^T.$$

The entries of Λ are the eigenvalues of A, and the columns of Q are the eigenvectors of A.

Characterizing definite matrices

A symmetric matrix A is:

- Positive definite iff all eigenvalues of A are positive
- Positive semidefinite iff all eigenvalues of A are nonnegative
- Negative definite iff all eigenvalues of A are negative
- Negative semidefinite iff all eigenvalues of A are nonpositive
- Indefinite iff A has both a positive and a negative eigenvalue

Proposition

Let A be a symmetric $n \times n$ matrix. A is positive definite if and only if all eigenvalues of A are positive.

If A is positive definite:

- Let λ be an eigenvalue of A, with an eigenvector $\mathbf{v}$.
- Consider the product $\mathbf{v}^T A \mathbf{v}$.

Proposition

Let A be a symmetric $n \times n$ matrix. A is positive definite if and only if all eigenvalues of A are positive.

If A is positive definite:

- Let λ be an eigenvalue of A, with an eigenvector $\mathbf{v}$.
- Consider the product $\mathbf{v}^T A\mathbf{v}$.

If all eigenvalues of A are positive:

- Let $\mathbf{x} \neq 0$.
- Use the spectral theorem to express $\mathbf{x}^T A\mathbf{x}$ as a positive sum.

Saddle points

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function for which Hf is continuous, and let $\mathbf{x}^* \in \mathbb{R}^n$ be a critical point of f. If $Hf(\mathbf{x}^*)$ is indefinite, then $\mathbf{x}^*$ is neither a local minimizer nor a local maximizer; it is a saddle point of $\mathbf{x}^*$.