Brualdi Chapter 2

1-23

I made a Gradescope page for this class. You may submit your first assignment there, or you may submit it on paper. I plan to have all assignments submitted on Gradescope in the future.

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Definition

Let A be a set, and let U be a set containing A. Then the set $\overline{A} = U \setminus A$ is the *complement* of A with respect to U.

Theorem (Subtraction principle)

If U is a finite set and $A \subseteq U$, then $|\overline{A}| = |U| - |A|$.

Example

How many ways are there to choose k elements from $\{1, \ldots, n\}$ (with repetition allowed) so that some element appears at least twice?

Section 2.2

Definition

Let S be a set of n elements, and let $r \leq n$. An *r-permutation* of S is a tuple $(a_1, \ldots, a_r)$ such that $a_1, \ldots, a_r$ are distinct elements from S.

Definition

A *permutation* of S is an n-permutation of S.

An r-permutation is sometimes called a *linear* r-permutation.

Bijections

Definition

Let S and T be sets. A *bijection* $f : S \to T$ is a function that is both

- injective (one-to-one), and
- surjective (onto).

A bijection is a *one-to-one correspondence*. If there exists a bijection $f : S \to T$, then $|S| = |T|$.

Permutations as bijections

A permutation of S can be represented as a bijection $f : \{1, \ldots, n\} \to S$.

For example, let $S = \{2, 3, 5, 7\}$.

$$(3, 5, 7, 2)$$

Definition

Let $r \leq n$ be nonnegative integers. $P(n, r)$ is the number of r-permutations of $\{1, \ldots, n\}$.

Theorem

$$P(n, r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

Alternative notation: $P(n, r) = n^{\underline{r}}$.

Example

In baseball, a *batting order* is an ordered list of nine players. If a team has 15 players, how many batting orders are possible?

Suppose Alice and Bob are players on the team. How many batting orders exist in which Alice or Bob appears fourth?

Definition

Let S be a finite set. An *r-circular permutation* of S is an arrangement r distinct elements of S clockwise in a circle (with no beginning or end).

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Alternatively, an r-circular permutation of S is an r-permutation $(a_1, \ldots, a_r)$ of S with the rule:

$$(a_1, \ldots, a_r) = (a_2, \ldots, a_r, a_1)$$

$$= (a_3, \ldots, a_r, a_1, a_2) = \cdots .$$

Example

Suppose n children are arranged in a circle. How many circular permutations of the children are possible?

- Division principle
- A simpler way

Theorem

If S is a set of n elements, then the number of circular r-permutations of S is

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}.$$

Example

How many circular permutations of $\{1, \ldots, n\}$ exist in which 1 and 2 are not adjacent?

- Subtraction principle
- Multiplication principle

Proof of binomial coefficient formula

If $n \geq k \geq 0$ are integers, then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

$$\begin{aligned}\binom{n}{k} &= \frac{P(n,k)}{k!} \\ &= \frac{n!}{k!(n-k)!}.\end{aligned}$$

Theorem

If $n \geq k \geq 0$ are integers, then $\binom{n}{k} = \binom{n}{n-k}$.

This is clear from the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
We can also prove this with a bijection.