Math 484: Nonlinear programming

Chapter 1: Lecture 2

The goal of this lecture is to review some important properties of n-dimensional real space.

We write

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}.$$

$\mathbb{R}^n$ is a

- Vector space
  (We can apply linear transformations to $\mathbb{R}^n$)
- Metric space
  (We have a notion of distance in $\mathbb{R}^n$)
- Topological space
  (We have a notion of open and closed sets in $\mathbb{R}^n$)

Brief review of vector spaces

Recall that a vector space is a set of objects (called *vectors*) which is equipped with a field (in this case $\mathbb{R}$). Here are some important properties of vectors:

- The sum of two vectors is a vector.
- The product of a vector with a field element (scalar) is a vector.
- Vectors can be multiplied using an *inner product* to produce a scalar.

Brief review of matrices

Recall that a (real) matrix is an $n \times m$ grid of real numbers. We can interpret a row or column of a real matrix as a vector in $\mathbb{R}^m$ or $\mathbb{R}^n$.

We write A_{ij} for the row i, column j entry of a matrix A. If A is an $\ell \times n$ matrix and B is an $n \times m$ matrix, then the product matrix AB (which is size $\ell \times m$) can be expressed by the formula:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Another useful matrix formula

If $\mathbf{u}^T = [u_1, \ldots, u_n]$, $\mathbf{v}^T = [v_1, \ldots, v_m]$, and A is an $n \times m$ matrix, then

$$\mathbf{u}^T A \mathbf{v} = \sum_{i=1}^{n} \sum_{j=1}^{m} u_i A_{ij} v_j.$$

Brief introduction to inner products

An inner product is defined on a vector space in order to introduce a notion of geometric concepts, like distance and angle.

An *inner product* on a vector space V with a field $\mathbb{R}$ is a binary operation $\cdot : V \times V \to \mathbb{R}$ that satisfies the following properties:

- $(\alpha \mathbf{x}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z})$
- $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$
- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
- $\mathbf{x} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x} \in V$, and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x}$ is the zero vector.

Vector norms

An inner product allows us to define a *norm* on a vector space, which gives us a way of measuring how close a vector is to zero.

Formally, a *norm* is a function $\|\cdot\| : V \to \mathbb{R}$ defined $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$.

A norm satisfies the following properties:

- $\|\alpha\,\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (Triangle Inequality)
- $|\,\mathbf{x} \cdot \mathbf{y}\,| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (Cauchy-Schwarz)

Inner product space geometry

We define the *angle* between two vectors $\mathbf{x}$ and $\mathbf{y}$ with the formula

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta.$$

When using the standard inner product in $\mathbb{R}^n$, the angle θ is equal to the physical angle between $\mathbf{x}$ and $\mathbf{y}$ when they are represented as arrows.

What does it mean if $\mathbf{x} \cdot \mathbf{y} = 0$?

Topology in $\mathbb{R}^n$

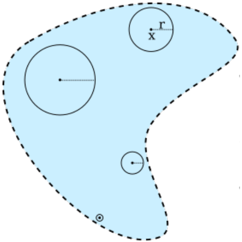
A *topology* on a set is a notion of *open* and *closed* subsets. We can define a topology on a set in order to use ideas like continuity, and to apply useful tools like the extreme value theorem.

Given a vector $\mathbf{x} \in \mathbb{R}^n$ and a value $r > 0$, the *open ball* of radius r around $\mathbf{x}$ is defined as

$$B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < r\}.$$

Topology in $\mathbb{R}^n$

We say that a subset $S \subseteq \mathbb{R}^n$ is *open* if for every point $\mathbf{x} \in S$ there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq S$.



Topology in $\mathbb{R}^n$

Given a subset $S \subseteq \mathbb{R}^n$, we say that $\mathbf{x} \in \mathbb{R}^n$ is a *boundary point* of S if for every $r > 0$,

$$S \cap B(\mathbf{x}, r) \neq \emptyset \text{ and}$$

$$(\mathbb{R}^n \setminus S) \cap B(\mathbf{x}, r) \neq \emptyset.$$

In other words, every open ball around $\mathbf{x}$, has a point inside S and a point outside S.

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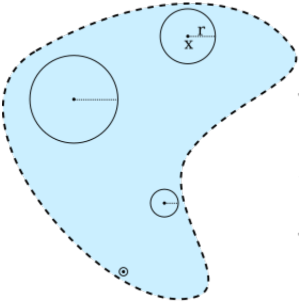
A subset $S \subseteq \mathbb{R}^n$ is open if and only if no point $\mathbf{x} \in S$ is a boundary point of S.

Topology in $\mathbb{R}^n$

We say that a subset $S \subseteq \mathbb{R}^n$ is *closed* if $\mathbb{R}^n \setminus S$ is open.



S is closed if and only if every boundary point of S belongs to S.

Topology in $\mathbb{R}^n$

For each of the following sets, decide whether the set is open, closed, both, or neither.

- $\mathbb{R}^n$ (taken as a subset of $\mathbb{R}^n$)
- $\emptyset$ (taken as a subset of $\mathbb{R}^n$)
- $(1, \infty)$ (taken as a subset of $\mathbb{R}$)
- $[1, \infty)$ (taken as a subset of $\mathbb{R}$)
- $\{\mathbf{0}\}$ (taken as a subset of $\mathbb{R}^n$)