Brualdi Chapter 2

1-25

This lecture will contain many proof techniques that can be used when dealing with binomial coefficients. This lecture is difficult, yet extremely important. It may take two days to complete.

Section 2.3

$\binom{n}{k}$ is the number of subsets of $\{1, \ldots, n\}$ of size k.

Alternatively, $\binom{n}{k}$ is the number of ways to choose a committee of k people from a group of n people.

If $k > n$, then $\binom{n}{k} = 0$.

Theorem

If $n \geq k \geq m \geq 0$ *are integers, then*

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Theorem

If $n \geq 0$ is an integer, then

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Definition

A *binary string* is a string that consists of 0's and 1's.

$\binom{n}{k}$ is the number of binary strings of length n with exactly k 1's.

Note: 2^n is the number of binary strings of length n. 2^n is also the number of subsets of a set of size n.

Theorem

For each integer $n \geq 0$,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

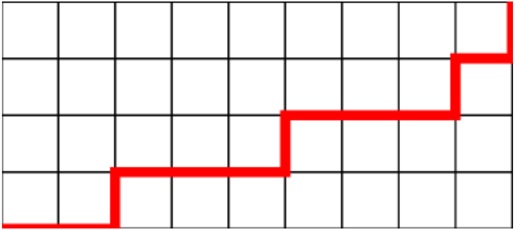
Theorem

If $n \geq r$ are integers, then

$$\binom{n+1}{r+1} = \sum_{k=0}^{n} \binom{k}{r}.$$

Definition

A *lattice path* is a path in the plane that only uses steps $\rightarrow$ and $\uparrow$.



Theorem

For integers $n \geq k \geq 0$, *the number of lattice paths from* $(0, 0)$ *to* $(n - k, k)$ *is* $\binom{n}{k}$.

Theorem

For integers $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Theorem

For integers $n \geq 0$,

$$\sum_{k=0}^{n}(k+1)\binom{2n-k}{n} = \binom{2n+2}{n+2}.$$

Theorem

Let $m \geq 0$ *be an integer. The number of sequences* $(a_1, \ldots, a_n)$ *such that* $a_1 \leq \cdots \leq a_n \leq m$ *is* $\binom{m+n}{n}$.